Supplementary Appendix
for The Origins of Top Firms

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S Supplementary Appendix

S.1 Nested CES Model

Alternatively, we can assume that each firm type $h \in \mathcal{H}$ produces a differentiated variety of the h-type good, where the elasticity of substitution across varieties is ν_h and across firm types is ν. We show that the steady-state equilibrium of this model is isomorphic to ours, with ν_h mapping into the returns to scale θ^h; however, these elasticities can be relevant for counterfactuals.

Demand. For the type-h good, the aggregator is

$$y_h = \left[\int \tilde{z}_i \, y_{hi}^{\frac{\nu_h - 1}{\nu_h}} \, di \right]^{\frac{\nu_h}{\nu_h - 1}},$$

where y_h is aggregate output of type h and y_{hi} is the output of firm i of type h. Across types, the aggregator is

$$Y = \left[\int \gamma_h^{\frac{1}{\nu}} \, y_h^{\frac{\nu - 1}{\nu}} \, dh \right]^{\frac{\nu}{\nu - 1}},$$

where Y is aggregate output. The corresponding price index for the type-h good is

$$P_h = \left[\int \tilde{z}_i^{\nu_h} \, p_{hi}^{1 - \nu_h} \, di \right]^{\frac{1}{1 - \nu_h}},$$

and the aggregate price index is

$$P = \left[\int \gamma_h \, P_h^{1 - \nu} \, dh \right]^{\frac{1}{1 - \nu}}.$$

Demand for each variety and the associated inverse demand are

$$y_{hi} = \gamma_h \, \tilde{z}_i^{\nu_h} \left(\frac{p_{hi}}{P_h} \right)^{-\nu_h} \left(\frac{P_h}{P} \right)^{-\nu} Y,$$

$$p_{hi} = \tilde{z}_i \, P_h^{1 - \frac{\nu}{\nu_h}} \, P^{\frac{\nu}{\nu_h}} \left(\frac{y_{hi}}{\gamma_h Y} \right)^{-\frac{1}{\nu_h}}.$$

We take aggregate output as the numeraire, so that $P = 1$.

Supply. The firm chooses k and l to solve

$$\max_{k \in [\underline{k}^h, \bar{k}^h],\, l} \; p_{hi}\, y_{hi} - wl - Rk - c_F,$$

where p_{hi} is given by the inverse demand above and the production function is $y_{hi} = \hat{z}_i\, f^h(k_i, l_i)$, with $\hat{\theta}^h$ denoting its returns to scale. Using the inverse demand, we can write revenue as

$$\begin{aligned} p_{hi}\, y_{hi} &= P_h^{\left(1-\frac{\nu}{\nu_h}\right)} \, (Y\,\gamma_h)^{\frac{1}{\nu_h}} \;\tilde{z}_i\, y_{hi}^{\frac{\nu_h-1}{\nu_h}} \\ &= \underbrace{P_h^{\left(1-\frac{\nu}{\nu_h}\right)} \, (Y\,\gamma_h)^{\frac{1}{\nu_h}}}_{\text{aggregate}} \;\; \underbrace{\tilde{z}_i\, \hat{z}_i^{\frac{\nu_h-1}{\nu_h}}}_{\text{TFP}} \;\; \underbrace{f^h(k_i, l_i)^{\frac{\nu_h-1}{\nu_h}}}_{\text{input aggregator}} . \end{aligned}$$

The TFP term can therefore be determined by demand or supply, and the effective returns to scale, $\frac{\nu_h-1}{\nu_h}\,\hat{\theta}^h$, reflect both the curvature of the demand function and the technology.

Equilibrium and counterfactuals. Our baseline steady-state equilibrium can be mapped into this model. Given the equilibrium aggregates Y and $\{y_h\}_{h\in\mathcal{H}}$, and using the definition of the aggregate price index P, we choose the demand weights $\{\gamma_h\}_{h\in\mathcal{H}}$ so that the aggregate component of revenue is equal to one for all type-h firms. The steady-state model is then isomorphic to ours when

$$\theta^h = \frac{\nu_h - 1}{\nu_h}\,\hat{\theta}^h \qquad \text{and} \qquad \exp\left(u_i^h + z_i\right) = \tilde{z}_i\, \hat{z}_i^{\frac{\nu_h-1}{\nu_h}} .$$

Importantly, the two models are not equivalent across counterfactuals, since the elasticities $\{\nu_h, \nu\}$ can affect relative prices across type-h goods. In our baseline model, these relative prices remain fixed.

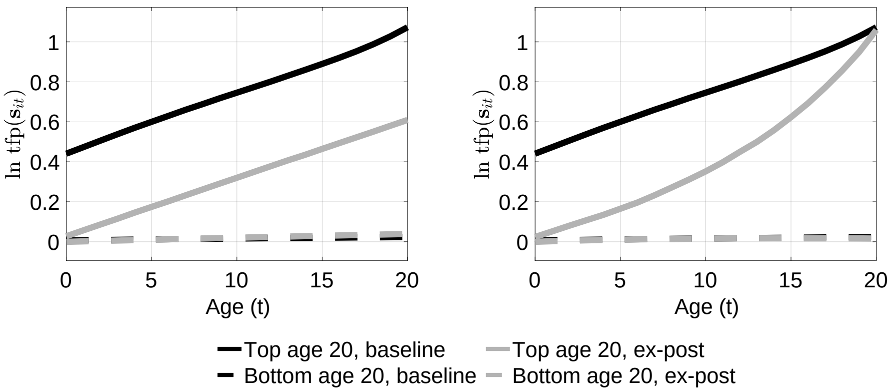
S.2 Ex-Ante Heterogeneity and Top and Bottom Firms Growth

To provide intuition on how the initial level and growth profile of top and bottom firms inform the ex-ante heterogeneity component of our baseline TFP process, we present a simple example comparing a simplified version of our TFP process with a standard TFP process that includes only ex-post shocks. Figure S.1 shows the TFP life cycle path for our baseline TFP process without entry and exit selection and no within type heterogeneity in the ex-ante component ($\underline{\sigma}_u = 0$) and a TFP process without the ex-ante component, i.e., $\exp(z_{it})$ with $z_{it+1} = (1-\rho_z)\mu_z + \rho_z z_{it} + \varepsilon_{it+1}$. For the baseline TFP process we use the parameters of the TFP process and the measure of potential entrants $\{\mathcal{E}_1, \mathcal{E}_2\}$ calibrated in Section 6, to match the observed initial output level and growth profile of top and bottom firms.

Panel (a) compares the simplified baseline TFP process with a process that excludes the ex-ante component, calibrated to roughly match the life-cycle profile of bottom firms and the growth profile of top firms (i.e., slope) implied by our baseline process. To match the

Figure S.1: TFP Process: Baseline vs. Ex-Post Heterogeneity

(a) Growth Profile of Top **(b)** Top Level at Age 20



Notes: The figure shows the average life-cycle profile of total factor productivity (TFP) for top 1% and bottom 99% firms at age 20 in the baseline model–assuming no entry and exit selection margins and no ex-ante within-type heterogeneity (i.e., $\underline{\sigma}_u = 0$)–with top and bottom firms represented by black solid and dashed lines, respectively. These profiles are compared to those obtained when the TFP process includes an ex-ante component $\exp(z_{it})$ with $z_{it+1} = (1-\rho_z)\mu_z + \rho_z z_{it} + \varepsilon_{it+1}$, shown by gray solid and dashed lines, respectively. Panel (a) shows the life-cycle profiles for the baseline TFP process and an alternative TFP process where the ex-post shocks have high persistence and low volatility $(\rho_z = 0.99, \sigma_z = 0.05, \mu_z = 0.3)$ such that it matches the bottom firms life cycle profile and growth profile of top firms (i.e., slope). Panel (b) shows the life-cycle profiles for the baseline TFP process and an alternative TFP process where the ex-post shocks have low persistence and high volatility $(\rho_z = 0.90, \sigma_z = 0.17, \mu_z = 0.03)$, such that it matches the bottom firms life cycle profile and the top firms' TFP level at age 20.

top firms' growth profile, we need an extremely persistent shock ($\rho_z = 0.99$) and very low volatility ($\sigma_z = 0.05$). Importantly, even if this unusual calibration allows us to match the growth profile of top firms, it remains far from fitting its level.

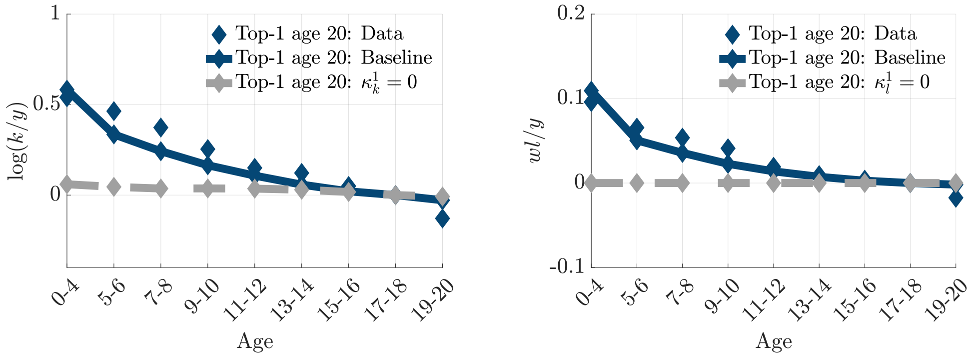
On the other hand, in Panel (b), we calibrate the ex-post process to fit the life-cycle profile of bottom firms and the TFP level of top firms at age 20 implied by our baseline process. This can be achieved with a relatively standard persistence ($\rho_z = 0.9$) and a relatively high volatility ($\sigma_z = 0.17$). In this case, the TFP process with only ex-post shocks implies an extremely high growth profile for top firms at age 20, as the initial shocks dissipate over time, causing the initial size of top firms at age 20 to be close to the average.

These examples illustrate that ex-ante heterogeneity in our model is informed by both the initial levels and growth profiles of top and bottom firms.

S.3 Input Usage Over The Life Cycle: Quantitative Model

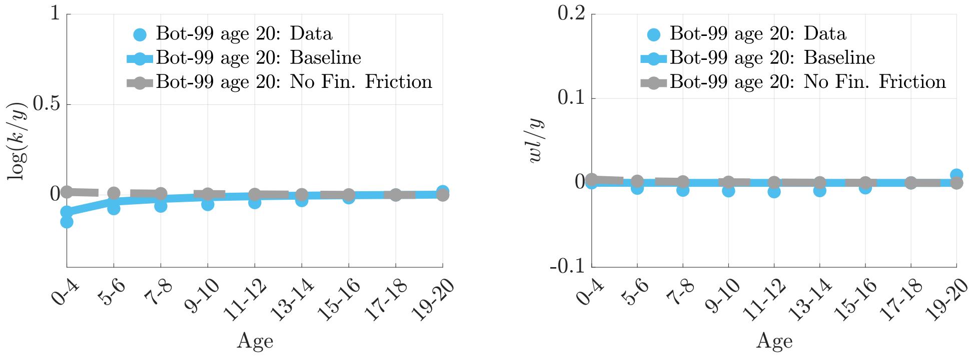
In this appendix, we study the determinants of input usage profiles among top and bottom firms over their life cycle in the quantitative model. Figure S.2 illustrates the key factors driving the life-cycle profiles of top and bottom firms in the model. In the absence of input-specific fixed costs, top firms' capital-output ratio (Figure S.2a) and labor share (Figure S.2b) would be mostly flat over their life cycle, which would be at odds with the data. On the other hand, in the absence of financing frictions, the capital-output ratio of bottom firms would be flat over their life cycle (Figure S.2c). In contrast, it is increasing in the data and the baseline model. Finally, financing frictions do not affect the labor share for bottom firms (Figure S.2d) because the capital-labor substitution, σ, is equal to 1. Thus, consistent with the qualitative analysis in Section 5, our quantitative model requires that top firms' technologies feature high input-specific fixed costs in capital and labor. In contrast, the bottom technologies' capital-output ratio is initially low due to financial frictions.

Figure S.2: The Role of Input-Specific Fixed Costs and Financial Frictions



(a) Top k/y **(b)** Top wl/y

(c) Bottom k/y **(d)** Bottom wl/y



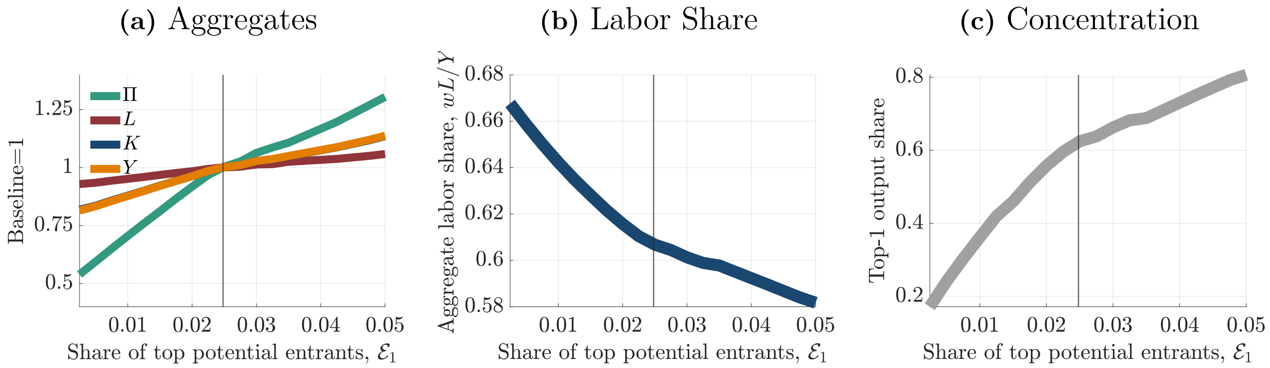
Notes:The figure shows the life cycle pattern of input usage, capital and labor, for top and bottom firms, respectively, in the data (dots) and using the model-simulated data. Panels (a) and (b) show input use among top firms when setting $\kappa_k^1 = 0$ and $\kappa_l^1 = 0$ (gray lines). The trajectory is in logs and is normalized to equal 0 at age 20. Panels (c) and (d) show input use among bottom firms in the absence of financial frictions (gray lines). The trajectory is in percentage points and it is normalized to 0 at age group 17-18.

S.4 Technological Frontier Changes

In this appendix, we analyze how exogenous changes to the technological frontier—specifically, the distribution of potential entrant measures $\{\mathcal{E}_h\}_{h\in\mathcal{H}}$—affect aggregate outcomes. Holding the total measure of potential entrants $\mathcal{E}$ fixed, we vary the share that are top technologies ($\mathcal{E}_1$). A greater availability of top technologies can be linked, for example, to better diffusion of ideas, both of which have been argued to be key drivers of economic growth (see, e.g., Lucas and Moll (2014) and Perla and Tonetti (2014)).

Figure S.3a shows how aggregate output changes with the availability of top technologies in the economy, i.e., as $\mathcal{E}_1$ varies. As expected, greater availability of high-growth technologies translates into higher aggregate output, capital, labor, and profits. Due to the change in the technology mix, specifically a higher prevalence of top technologies, labor doesn't increase as much as output, which, in Figure S.3b, translates into a lower labor share as the availability of top technologies increases. Finally, Figure S.3c shows that output concentration increases with the availability of top technologies in the economy.

Figure S.3: Technological Frontier Changes



(a) Aggregates **(b)** Labor Share **(c)** Concentration

Notes: Steady state comparisons in GE solving for new equilibrium wage w. Aggregate variables in Panel (a) are normalized relative to the baseline economy.

In summary, these results show that if the economy grows due to greater availability of top technologies, our model predicts increased concentration in economic activity, lower labor shares, and higher profit shares, even without firms gaining greater market power.

S.5 Subsidy to Required Capital

We assume that the government subsidizes a fraction $s_\kappa \in [0,1]$ of the capital requirements $\{\kappa_k^h\}$ and finances the subsidy with a lump-sum tax T on all operating firms. The government budget constraint is

$$Rs_\kappa \sum_{h\in\mathcal{H}} M^h \kappa_k^h = MT, \tag{1}$$

where $R = r + \delta$ is the rental rate, M^h is the measure of type-h firms, and $M = \sum_{h\in\mathcal{H}} M^h$ is the total measure of firms. Under this policy, a firm of type h receives $s_\kappa \kappa_k^h$ units of capital from the government and therefore needs to rent only $k - s_\kappa \kappa_k^h$ units of capital. This reduces the firm's rental costs and relaxes the feasibility constraint, which now requires

$$\bar{k} > (1 - s_\kappa)\kappa_k^h,$$

where $\bar{k}$ denotes the maximum amount of capital a firm can rent given its assets and value. In the limiting case in which $s_\kappa = 1$, the firm no longer faces the non-homothetic capital requirement, although it faces a higher equilibrium lump-sum tax T.

Intuitively, by lowering the amount of capital firms need to rent, the subsidy reduces rental costs and may relax financing frictions. At the same time, the lump-sum tax imposes an additional fixed cost on all operating firms, which may hurt firms with small upfront capital requirements, such as firms with bottom technologies.

Table S.1 shows how key aggregate outcomes vary with s_κ. The subsidy increases output per worker, mainly through an increase in TFP among top firms and a rise in their prevalence. These effects arise because top firms become less financially constrained and can overcome the feasibility constraint even when external financing is limited. At the same time, the subsidy raises wages and requires higher taxes, which affect firms with bottom technologies because they do not benefit from the subsidy. As the subsidy becomes larger, the tax burden on bottom firms becomes sizable, eventually preventing some of them from entering.

When the government sets $s_\kappa = 1$, the quantitative changes in output per worker, productivity, and wages are similar to those in the exercise that removes the non-homothetic capital input costs (see Section 7.1.2). This is consistent with our observation that the direct aggregate importance of these costs, through deadweight losses, is small. The main difference is that the tax, which amounts to 1% of GDP, imposes an additional fixed cost on bottom firms, making reallocation stronger than in the baseline exercise that removes the non-homothetic input costs. Thus, this policy roughly implements the counterfactual that removes the non-homothetic input costs in Section 7.1.2. It is worth noting that, although this policy has large benefits, a key challenge to its implementation is that firm types and their capital requirements are not directly observed.

Table S.1: Subsidy to Required Capital

	Subsidy rate: s_κ				
	0.0	0.1	0.2	0.5	1.0
Relative to Baseline (=1)					
Output per worker, Y/L	1.00	1.01	1.05	1.12	1.19
Capital-output, K/Y	1.00	1.00	1.00	1.01	1.02
TFP top-h	1.00	1.01	1.07	1.16	1.25
TFP bot-h	1.00	1.00	1.00	-	-
Wage, w	1.00	1.01	1.02	1.04	1.11
Top-h incumbents	1.00	1.10	1.46	2.35	3.82
Bot-h incumbents	1.00	1.01	0.89	0.00	0.00
Output share top-h	1.00	1.01	1.19	1.56	1.56
Fiscal Revenue$/Y$ (%)	0.0%	0.0%	0.1%	0.4%	1.0%

Notes: Results are relative to the baseline model, in which $s_\kappa = 0$. The ratio of total tax revenue to output, MT/Y, is reported in levels.

S.6 Alternative Theory for Top Firms' Declining Input Shares

To simplify the notation we drop i firm and t time subscripts, and h technology-type superscripts. We assume that the firm doesn't face financial frictions and the technology is homothetic, but their price is elastic $p(y)$ to output and wages are elastic $w(l)$ to labor, and input efficiency can change over time z^j for $j = \{k, l\}$. Then, the firms solve the profit maximization problem

$$\max_{y,l,k} p\left(y\right) y - w\left(l\right) l - Rk - c_F \tag{2}$$

subject to

$$\begin{aligned} y &= zx^{\theta} \\ x &= \left[\alpha^{\frac{1}{\sigma}} \left(z_k k\right)^{\frac{\sigma-1}{\sigma}} + \left(1-\alpha\right)^{\frac{1}{\sigma}} \left(z_l l\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}. \end{aligned}$$

The FOC of problem (2) are

$$\begin{aligned} [l]: &\quad \left[\frac{\partial p\left(y\right)/p\left(y\right)}{\partial y/y} + 1\right] \frac{\partial y}{\partial l} p\left(y\right) = \left[\frac{\partial w\left(l\right)/w\left(l\right)}{\partial l/l} + 1\right] w\left(l\right) \\ [k]: &\quad \left[\frac{\partial p\left(y\right)/p\left(y\right)}{\partial y/y} + 1\right] \frac{\partial y}{\partial k} p\left(y\right) = R. \end{aligned}$$

We denote $\mu_w = \frac{\partial w(l)/w(l)}{\partial l/l} + 1$ as the proportional shadow cost of labor (i.e., if $\mu_w = 1$, there is no shadow cost), and $\frac{\partial p(y)/p(y)}{\partial y/y} + 1 = \frac{1}{\mu_y}$, where μ_y represents the markup (i.e., if $\mu_y = 1$ there is no markup). Thus, the FOC can be rewritten as

$$\begin{aligned} \frac{1}{\mu_y}\theta\left(1-\alpha\right)^{\frac{1}{\sigma}} z_l \frac{\left(z_l l\right)^{-\frac{1}{\sigma}}}{x^{\frac{\sigma-1}{\sigma}}} yp\left(y\right) &= \mu_w w\left(l\right) \\ \frac{1}{\mu_y}\theta\alpha^{\frac{1}{\sigma}} z_k \frac{\left(z_k k\right)^{-\frac{1}{\sigma}}}{x^{\frac{\sigma-1}{\sigma}}} yp\left(y\right) &= R. \end{aligned}$$

Combining the FOC we get the relationship

$$\frac{z_k k}{z_l l} = \frac{\alpha}{1-\alpha}\left(\frac{\mu_w w\left(l\right)}{R}\frac{z_k}{z_l}\right)^{\sigma}.$$

Again doing some algebra and combining with the FOC we find

$$\begin{aligned} \frac{w\left(l\right) l}{p\left(y\right) y} &= \frac{1}{\mu_y}\theta\left(1-\alpha\right)^{\frac{1}{\sigma}}\left[\alpha^{\frac{1}{\sigma}}\left(\frac{z_k k}{z_l l}\right)^{\frac{\sigma-1}{\sigma}} + \left(1-\alpha\right)^{\frac{1}{\sigma}}\right]^{-1} - \frac{\left(\mu_w - 1\right) w\left(l\right) l}{p\left(y\right) y} \\ \frac{k}{p\left(y\right) y} &= \frac{1}{\mu_y}\theta\alpha^{\frac{1}{\sigma}}\frac{1}{R}\left[\alpha^{\frac{1}{\sigma}} + \left(1-\alpha\right)^{\frac{1}{\sigma}}\left(\frac{z_l l}{z_k k}\right)^{\frac{\sigma-1}{\sigma}}\right]^{-1}. \end{aligned}$$

Combining the previous three equations, we get the wl/py, k/py, and k/wl ratios:

$$\frac{w\left(l\right)l}{p\left(y\right)y} = \theta\left(1-\alpha\right)\frac{1}{\mu_y}\left[\alpha\left(\frac{R}{\mu_w w\left(l\right)}\right)^{1-\sigma}\left(\frac{z_l}{z_k}\right)^{1-\sigma}+\left(1-\alpha\right)\right]^{-1} - \frac{\left(\mu_w-1\right)w\left(l\right)l}{p\left(y\right)y} \tag{3}$$

$$\frac{k}{p\left(y\right)y} = \frac{\theta\alpha}{R}\frac{1}{\mu^y}\left[\alpha+\left(1-\alpha\right)\left(\frac{\mu_w w\left(l\right)}{R}\right)^{1-\sigma}\left(\frac{z_k}{z_l}\right)^{1-\sigma}\right]^{-1} \tag{4}$$

$$\frac{k}{w\left(l\right)l} = \left(\frac{\alpha}{1-\alpha}\right)\left(\frac{z_l}{z_k}\right)^{1-\sigma}\left(\frac{\mu_w}{R}\right)^{\sigma}\left(w\left(l\right)\right)^{\sigma-1} \tag{5}$$

S.7 Non-Homothetic CES

We show that a non-homothetic CES production technology cannot, by itself, generate the empirical patterns we document for top firms in Section 3. As the firm grows, it reallocates expenditure toward the highest-income-elasticity input if inputs are complements (or toward the lowest-income-elasticity input if they are substitutes), so both factor shares cannot decline simultaneously under this standard technology specification.

Technology. The firm produces output y using capital k and labor l according to

$$y = z\left[\gamma_k^{1/\sigma}\left(\frac{k}{y^{\epsilon_k-1}}\right)^{(\sigma-1)/\sigma}+\gamma_l^{1/\sigma}\left(\frac{l}{y^{\epsilon_l-1}}\right)^{(\sigma-1)/\sigma}\right]^{\alpha\sigma/(\sigma-1)},$$

where z is a productivity shifter, γ_k and γ_l are input weights, σ is the elasticity of substitution between capital and labor, ϵ_k, ϵ_l are the income elasticities of capital and labor, and $\alpha \in (0,1]$ imposes additional curvature, respectively.

Firm problem. Let R denote the rental rate of capital and w the wage. The firm solves

$$\pi = \max_{k,l}\ y - Rk - wl \quad \text{s.t.} \quad y = z\left[\gamma_k^{1/\sigma}\left(\frac{k}{y^{\epsilon_k-1}}\right)^{(\sigma-1)/\sigma}+\gamma_l^{1/\sigma}\left(\frac{l}{y^{\epsilon_l-1}}\right)^{(\sigma-1)/\sigma}\right]^{\alpha\sigma/(\sigma-1)}.$$

We derive cost shares below.

Input demand. From cost minimization $\min_{k,l}\ Rk+wl$ subject to the production constraint, the first-order conditions yield, for $i \in \{k,l\}$,

$$p_i x_i = \mu\, z\, \alpha\, A^{\alpha\sigma/(\sigma-1)-1}\,\gamma_i^{1/\sigma}\left(\frac{x_i}{y^{\epsilon_i-1}}\right)^{(\sigma-1)/\sigma},$$

where μ is the Lagrange multiplier and A is the inner aggregator. Taking the ratio of FOCs gives relative demand

$$\frac{k}{l} = \frac{\gamma_k}{\gamma_l}\left(\frac{R}{w}\right)^{-\sigma} y^{(1-\sigma)(\epsilon_k-\epsilon_l)}.$$

Substituting back into the aggregator yields the Hicksian demands

$$k = \left(\frac{y}{z}\right)^{1/\alpha} y^{(1-\sigma)\epsilon_k - 1}\, \gamma_k \left(\frac{R}{P(y;R,w)}\right)^{-\sigma}, \qquad l = \left(\frac{y}{z}\right)^{1/\alpha} y^{(1-\sigma)\epsilon_l - 1}\, \gamma_l \left(\frac{w}{P(y;R,w)}\right)^{-\sigma},$$

where we define the non-homothetic input cost aggregator

$$P(y;R,w) \equiv \left[\gamma_k\, R^{1-\sigma}\, y^{\epsilon_k(1-\sigma)} + \gamma_l\, w^{1-\sigma}\, y^{\epsilon_l(1-\sigma)}\right]^{1/(1-\sigma)}.$$

Total variable cost is then

$$Rk + wl = P(y;R,w)\, z^{-1/\alpha}\, y^{1/\alpha - 1}.$$

Cost and profit shares. Define the CES-style shares

$$s_k(y) \equiv \frac{\gamma_k\, R^{1-\sigma}\, y^{\epsilon_k(1-\sigma)}}{\gamma_k\, R^{1-\sigma}\, y^{\epsilon_k(1-\sigma)} + \gamma_l\, w^{1-\sigma}\, y^{\epsilon_l(1-\sigma)}}, \qquad s_l(y) = 1 - s_k(y),$$

and the share-weighted average income elasticity $\bar{\epsilon}(y) \equiv s_k\epsilon_k + s_l\epsilon_l$. The profit-maximizing condition $\mathrm{MC} = 1$ takes the form

$$\mathrm{MC}(y) = \left(\frac{1}{\alpha} - 1 + \bar{\epsilon}(y)\right)\frac{Rk + wl}{y} = 1,$$

which yields the cost and profit shares at the optimum:

$$\frac{Rk}{y} = \frac{s_k}{\frac{1}{\alpha} - 1 + \bar{\epsilon}(y)} \tag{6}$$

$$\frac{wl}{y} = \frac{s_l}{\frac{1}{\alpha} - 1 + \bar{\epsilon}(y)} \tag{7}$$

$$\frac{\pi}{y} = 1 - \frac{1}{\frac{1}{\alpha} - 1 + \bar{\epsilon}(y)}. \tag{8}$$

Each factor share is the standard CES share adjusted by the non-homotheticity wedge in the denominator.

Asymptotics. Assume $\sigma < 1$ (capital and labor are gross complements) and, without loss of generality, $\epsilon_l > \epsilon_k$ (labor is the more income-elastic input). As $y \to \infty$,

$$\lim_{y\to\infty} s_l = \lim_{y\to\infty} \frac{\gamma_l\, w^{1-\sigma}}{\gamma_l\, w^{1-\sigma} + \gamma_k\, R^{1-\sigma}\, y^{(\epsilon_k - \epsilon_l)(1-\sigma)}} = 1, \qquad \lim_{y\to\infty} s_k = 0.$$

The asymptotic shares are therefore

$$\frac{wl}{y} \to \frac{1}{\frac{1}{\alpha} - 1 + \epsilon_l}, \qquad \frac{Rk}{y} \to 0, \qquad \frac{\pi}{y} \to 1 - \frac{1}{\frac{1}{\alpha} - 1 + \epsilon_l}.$$

At scale, only the higher-ϵ input (labor, under our assumption) retains a positive expenditure share, while capital's share vanishes. Since $\epsilon_l \geq \bar{\epsilon}(y)$ for all y, the profit share is back-loaded along the firm's growth path.

Comparative statics of input shares. We now examine how factor shares evolve with size y. With $\epsilon_l > \epsilon_k$, labor's share is

$$\frac{wl}{y} = \frac{s_l}{\frac{1}{\alpha} - 1 + s_l(\epsilon_l - \epsilon_k) + \epsilon_k},$$

where, rearranging,

$$s_l = \frac{\gamma_l\, w^{1-\sigma}}{\gamma_l\, w^{1-\sigma} + \gamma_k\, R^{1-\sigma}\, y^{-(\epsilon_l - \epsilon_k)(1-\sigma)}}.$$

With $\sigma < 1$ and $\epsilon_l > \epsilon_k$, the second denominator term shrinks with y, so s_l is increasing in y. Differentiating labor's cost share with respect to s_l,

$$\frac{\partial(wl/y)}{\partial s_l} = \frac{\frac{1}{\alpha} - 1 + \epsilon_k}{\left[\frac{1}{\alpha} - 1 + s_l(\epsilon_l - \epsilon_k) + \epsilon_k\right]^2} > 0,$$

which is positive whenever $\alpha < 1$ (so $1/\alpha - 1 > 0$) and $\epsilon_k \geq 0$. Combined with s_l rising in y, this implies wl/y is increasing in y. An analogous argument shows Rk/y is decreasing in y, and the residual π/y is increasing in y.